HARBIN ENGINEERING UNIVERSITY

Integration Techniques Review

[quant67]

November 2, 2014

1 Basic Integration Formulas

1. $\int du = u + C$
2. $\int k du = ku + C$
3. $(du + dv) = \int du + \int dv$
4. $\int u^n du = \frac{u^{n+1}}{n+1} + C \quad (u \neq -1)$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int \sin u du = -\cos u + C$
7. $\int \cos u du = \sin u + C$
8. $\int \sec^2 u du = \tan u + C$
9. $\int \csc^2 u du = -\cot u + C$
10. $\int \sec u \tan u du = \sec u + C$
11. $\int \csc u \cot u du = -\csc u + C$
12. $\int \tan u du = -\ln|\cos u| + C$
13. $\int \cot u du = \ln|\sin u| + C$
14. $\int e^u du = e^u + C$
15. $\int a^u du = \frac{a^u}{\ln a} + C$
16. $\int \sinh u du = \cosh u + C$
17. $\int \cosh u du = \sinh u + C$
18. $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
19. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
20. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$
21. $\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \quad (a > 0)$
22. $\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C (u > a > 0)$

1.1 Making a Simplifying Substitution

Evaluate

$$\int \frac{2x-9}{\sqrt{x^2-9x+1}} dx.$$

Solution

$$\begin{aligned}
\int \frac{2x-9}{\sqrt{x^2-9x+1}} dx &= \int \frac{du}{\sqrt{u}} \\
&= \int u^{-1/2} du \\
&= \frac{u^{(-1/2)+1}}{(-1/2)+1} + C \\
&= 2u^{1/2} + C \\
&= 2\sqrt{x^2-9x+1} + C
\end{aligned}$$

1.2 Completing the Square

Evaluate

$$\int \frac{dx}{\sqrt{8x-x^2}}.$$

Solution We complete the square to write the radicand as

$$\begin{aligned} 8x-x^2 &= -(x^2-8x) = -(x^2-8x+16-16) \\ &= -(x^2-8x+16)+16 = 16-(x-4)^2 \end{aligned}$$

Then

$$\begin{aligned} \int \frac{dx}{\sqrt{8x-x^2}} &= \int \frac{dx}{\sqrt{16-(x-4)^2}} \\ &= \int \frac{du}{a^2-u^2} \\ &= \sin^{-1}\left(\frac{u}{a}\right)+C \\ &= \sin^{-1}\left(\frac{x-4}{4}\right)+C. \end{aligned}$$

1.3 Expanding a Power and Using a Trigonometric Identity

Evaluate

$$\int (\sec x+\tan x)^2 dx$$

Solution We expand the integrand and get

$$(\sec x+\tan x)^2 = \sec^2 x+2\sec x\tan x+\tan^2 x.$$

We replace $\tan^2 x$ by $\sec^2 -1$ and get

$$\begin{aligned} \int (\sec x+\tan x)^2 dx &= \int (\sec^2 x+2\sec x\tan x+\sec^2 x-1)dx \\ &= 2\int \sec^2 x dx+2\int \sec x\tan x dx-\int 1 dx \\ &= 2\tan x+2\sec x-x+C. \end{aligned}$$

1.4 Eliminating a Square Root

Evaluate

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\,dx.$$

Solution We use the identity

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \qquad or \qquad 1+\cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this identity becomes

$$1+\cos 4x = 2\cos^2 2x.$$

Hence

$$\begin{aligned}
\int_0^{\pi/4} \sqrt{1+\cos 4x}\,dx &= \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\,dx \\
&= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\,dx \\
&= \sqrt{2}\int_0^{\pi/4} \cos 2x\,dx \\
&= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} \\
&= \frac{\sqrt{2}}{2}.
\end{aligned}$$

1.5 Reducing am Improper Fraction

Evaluate

$$\int \frac{3x^2-7x}{3x+2}dx.$$

Solution The integrand is an improper fraction (degree of numerator greater then or equal to degree of denominator). To integrate it, we divide first, getting a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2-7x}{3x+2} = x-3+\frac{6}{3x+2}.$$

Therefore

$$\int \frac{3x^2-7x}{3x+2}dx = \int\left(x-3+\frac{6}{3x+2}\right)dx = \frac{x^2}{2}-3x+2\ln|3x+2|+C.$$

1.6 Separating a Fraction

Evaluate

$$\int \frac{3x+2}{\sqrt{1-x^2}}dx.$$

Solution We first separate the integrand to get

$$\int \frac{3x+2}{\sqrt{1-x^2}}dx = 3\int \frac{xdx}{\sqrt{1-x^2}} + 2\int \frac{dx}{\sqrt{1-x^2}}.$$

In the first of these new integrals, we substitute

$$u = 1-x^2, \qquad du = -2xdx, \qquad and \qquad xdx = -\frac{1}{2}du.$$

$$3\int \frac{xdx}{\sqrt{1-x^2}} = 3\int \frac{(-1/2)du}{\sqrt{u}} = -\frac{3}{2}\int u^{-1/2}du = -3\sqrt{1-x^2}+C.$$

1.7 Multiplying by a Form of 1

Evaluate

$$\int \sec x dx.$$

Solution

$$\begin{aligned}\int \sec x dx &= \int (\sec x)(1) dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} dx \\ &= \int \sec^2 x + \sec x \tan x \sec x + \tan x dx \\ &= \int \frac{du}{u} \\ &= \ln|u| + C = \ln|\sec x + \tan x| + C.\end{aligned}$$

With cosecants and cotangents in place of secants and tangents, the method above lead to a pompanion formula for the integral of the cosecant.

$$\int \csc u du = -\ln|\csc u + \cot u| + C.$$

2 Integration by Parts

Integration by Part Formula

$$\int u dv = uv - \int v du$$

2.1 Using Integration by Parts

Evaluate

$$\int x \cos x dx.$$

Solution We use the formula

$$\int u dv = uv - \int v du$$

with

$$u = x, \quad dv = \cos x dx.$$

Than

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + C.$$

2.2 Finding area

Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.
Solution The region's area is

$$\int_0^4 xe^{-x} dx.$$

We use the formula $\int u dv = uv - \int v du$ with

$$u = x, \quad dv = e^{-x} dx, \quad du = dx, \quad v = e^{-x}.$$

Then

$$\begin{aligned}\int xe^{-x} dx &= -xe^{-x} - \int (-e^{-x}) dx \\ &= -xe^{-x} + \int e^{-x} dx \\ &= -xe^{-x} - e^{-x} + C.\end{aligned}$$

Then

$$\begin{aligned}\int_0^4 xe^{-x} dx &= \left[-xe^{-x} - e^{-x}\right]_0^4 \\ &= 1 - 5e^{-4} \approx 0.91.\end{aligned}$$

2.3 Integral of the Natural Logarithm

Find

$$\int \ln x dx.$$

Solution Since $\int \ln x$ can be written as $\int \ln x \cdot 1 dx$, we use the formula $\int u dv = uv - \int v du$ with

$$u = \ln x \qquad v = x.$$

Then

$$\int \ln x dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C$$

2.4 Repeated Use of Integration by Parts

Evaluate

$$\int x^2 e^x dx.$$

Solution With $u = x^2$ and $v = e^x$, we have

$$\int x^2 e^x dx = x^2 e^x - 2 \int xe^x dx.$$

Then

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C.$$

with $u = x, v = e^x$.
Hence,

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + C.$$

2.5 Solving for the Unknown Integral

Evaluate

$$\int e^x \cos x dx.$$

Solution Let $u = e^x$ and $dv = \cos x dx$ then

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x dx.$$

Then

$$\begin{aligned} \int e^x \cos x dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x dx)\right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x dx. \end{aligned}$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides gives

$$2\int e^x \cos x dx = e^x \sin x + e^x \cos x + C.$$

Dividing by 2 and renaming the constant of integration gives

$$\int e^x \cos x dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

2.6 Using Tabular Integration

Evaluate

$$\int x^2 e^x dx.$$

Solution with $f(x) = x^2$ *and* $g(x) = e^x$, we list:

$f(x)$ and its derivatives		**$g(x)$ and its integrals**
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x.

We combine the products of the functions connected by the arrows acording to the operation signs above the arrows to obtain

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + C.$$

2.7 Using Tabular Integration

Evaluate

$$\int x^3 \sin x dx.$$

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		**$g(x)$ and its integrals**
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x.$

Again we combine the products of the functions connected by the arrows according tho the operation signs above the arrows to obtain

$$\int x^3 \sin x dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6\sin x + C.$$

3 partial Fractions

Method of Partial Fractions ($f(x)/g(x)$) Proper

Step 1: Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r^2)} + \cdots + \frac{A_m}{(x-r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

Step 2: Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1 x + C_1}{x^2 + px + 1} + \frac{B_2 x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_n x + C_n}{(x^2 + px + q)^2}.$$

Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.
Step 3: Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.
Step 4: Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

3.1 Using a Repeated Linear Factor

Express as a sum of partial fractions:

$$\frac{6x+7}{(x+2)^2}.$$

Solution According to the description above, we must express the fraction as a sum of partial fractions with undetermined coefficients.

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}.$$

$$\begin{aligned} 6x+7 &= A(x+2)+B \\ &= Ax+(2A+B) \end{aligned}$$

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad and \quad 2A+B = 7 \quad or \quad A = 6 \quad and \quad B = -5.$$

Therefore,

$$\frac{6x+7}{(x+2)^2} = \frac{6}{x+2} - \frac{5}{(x+2)^2}.$$

3.2 Integrating with an Irreducible Quadratic Factor in the Denominator

Evaluate

$$\int \frac{-2x+4}{(x^2+1)(x-1)}$$

using partial fractions.
Solution The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x+4}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}.$$

Clearing the equation of fractions gives

$$A = 2, \quad B = 1, \quad C = -1, \quad D = 1.$$

And

$$\begin{aligned} \int \frac{-2x+4}{(x^2+1)(x-1)} &= \int \left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2} \right) dx \\ &= \ln(x^2+1) + \tan^{-1} x - 2\ln|x-1| - \frac{1}{x-1} + C. \end{aligned}$$

4 Trigonometric Substitutions

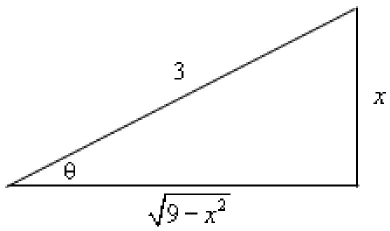
Trigonometric substitutions enable us to replace the binomials $a^2+x^2, a^2-x^2, \quad and \quad x^2-a^2$ by single squared terms and thereby transform a number of integrals containing square roots into integrals we can evaluate directly.

4.1 Using the Substitution $x = a\sin\theta$

Evaluate

$$\int \frac{x^3 dx}{\sqrt{9-x^2}}, \qquad -3 < x < 3$$

Solution We set



$$x = 3\sin\theta, \qquad dx = 3\cos\theta d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$9 - x^2 = 9 - 9\sin^2\theta = 9\cos^2\theta.$$

Then

$$\begin{aligned}\int \frac{x^3 dx}{\sqrt{9-x^2}} &= \int \frac{27\sin^3\theta \cdot 3\cos\theta d\theta}{|3\cos\theta|} \\ &= 27\int sin^3\theta \qquad\qquad \cos\theta > 0 \quad for \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ &= -9\sqrt{9-x^2} + \frac{(9-x^2)^{3/2}}{3} + C.\end{aligned}$$

5 L'Hôpital's Rule

Theorem 1 L'Hôpital's Rule
Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and $g'(a) \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$